Step III, Solutions
June 2007

Section A: Pure Mathematics

1. The first result can be obtained by applying a compound angle formula to $\tan((\theta_1+\theta_2)+(\theta_3+\theta_4))$ and then repeating the application to each of $\tan(\theta_1+\theta_2)$ and $\tan(\theta_3+\theta_4)$ where they appear. On simplification, this gives

$$\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{t_1+t_2+t_3+t_4-t_2t_3t_4-t_3t_4t_1-t_4t_1t_2-t_1t_2t_3}{1-t_1t_2-t_1t_3-t_1t_4-t_2t_3-t_2t_4-t_3t_4+t_1t_2t_3t_4}.$$

As t_1, etc are the roots of the equation $at^4+bt^3+ct^2+dt+e=0$, then $at^4+bt^3+ct^2+dt+e=a(t-t_1)(t-t_2)(t-t_3)(t-t_4)$, which yields, from expansion and comparison of coefficients, the four results

$t_1+t_2+t_3+t_4=\frac{-b}{a}$, $t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=\frac{c}{a}$,

$t_2t_3t_4+t_3t_4t_1+t_4t_1t_2+t_1t_2t_3=\frac{-d}{a}$, and $t_1t_2t_3t_4=\frac{e}{a}$.

These substituted in the first result lead to $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{-b+d}{a-c+e}$.

Applying double and compound angle formulae to $p\cos 2\vartheta+\cos(\vartheta-\alpha)+p=0$ gives the equation $2p\cos^2\vartheta+\cos\vartheta\cos\alpha+\sin\vartheta\sin\alpha=0$, which can be rearranged as $\cos\alpha+\tan\vartheta\sin\alpha=\frac{-2p}{\sec\vartheta}$.

Squaring this and replacing $\tan\vartheta$ by t, $(\cos\alpha+t\sin\alpha)^2=\frac{4p^2}{1+t^2}$.

Rearranging this obtains the quartic equation $t^4\sin^2\alpha+t^3\sin 2\alpha+t^2+t\sin 2\alpha+(\cos^2\alpha-4p^2)=0$, and so, from the second result $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{0}{-4p^2}=0$, and thus $\theta_1+\theta_2+\theta_3+\theta_4=n\pi$.

2. (i)

$$1.3.5.7.....(2n-1)=\frac{1.2.3.4.....2n}{2.4.6.8.....2n}==\frac{(2n)!}{2.1.2.2.2.3.2.4.....2.n}=\frac{(2n)!}{2^n.1.2.3.4.....n}=\frac{(2n)!}{2^n n!}$$

Using the binomial theorem, which is valid given the condition $|x|<\frac{1}{4}$,

$$(1-4x)^{\frac{-1}{2}}=1+\frac{-1}{2}(-4x)+\frac{\frac{-1}{2}\frac{-3}{2}(-4x)^2}{2!}+...$$

$$=1+1.(2x)+\frac{1.3}{2!}(2x)^2+...+\frac{1.3.5.7...(2n-1)}{n!}(2x)^n+...$$

So the first result of the question yields $(1-4x)^{\frac{-1}{2}}=1+\sum_{n=1}^{\infty}\frac{\frac{(2n)!}{2^n n!}(2x)^n}{n!}$ leading to the required expression.

(ii) Differentiating $(1-4x)^{\frac{-1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n)!x^n}{(n!)^2}$ with respect to x, and multiplying the result by x gives $\frac{2x}{(1-4x)^{\frac{3}{2}}}=\sum_{n=1}^{\infty}\frac{(2n)!x^n}{n!(n-1)!}$ and substituting $x=\frac{6}{25}<\frac{1}{4}$, gives the desired result.

(iii) Integrating $(1-4x)^{\frac{-1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n)!x^n}{(n!)^2}$ with respect to x, gives $\frac{-1}{2}(1-4x)^{\frac{1}{2}}=x+\sum_{n=1}^{\infty}\frac{(2n)!x^{n+1}}{(n+1)!n!}+c$, and substituting $x=0<\frac{1}{4}$, gives $c=\frac{-1}{2}$.

Now substituting $x=\frac{2}{9}=\frac{2}{3^2}<\frac{1}{4}$ and simplifying, gives the desired result.

3. (i) $F_3=2, F_4=3, F_6=5, F_6=8, F_7=13, F_8=21$

(ii) The result requires no term beyond F_{2k+2} should appear on the RHS so the first strategy is to replace F_{2k+3} and hence

$$F_{2k+3}F_{2k+1}-F_{2k+2}{}^2=(F_{2k+2}+F_{2k+1})F_{2k+1}-F_{2k+2}{}^2=(F_{2k+1}-F_{2k+2})F_{2k+2}+F_{2k+1}{}^2=-F_{2k}F_{2k+2}+F_{2k+1}{}^2$$

as required.

(iii) The initial case is trivial to demonstrate, and so the induction runs from assuming that $F_{2k+1}F_{2k-1}-F_{2k}{}^2=1$, and attempting to prove that

$F_{2(k+1)+1}F_{2(k+1)-1}-F_{2(k+1)}{}^2=1$.

$F_{2(k+1)+1}F_{2(k+1)-1}-F_{2(k+1)}{}^2=F_{2k+3}F_{2k+1}-F_{2k+2}{}^2=-F_{2k}F_{2k+2}+F_{2k+1}{}^2$ from (ii)

$=-(-F_{2k-1}F_{2k+1}+F_{2k}{}^2)$ by a similar argument to (ii) $=-(-1)$ by inductive hypothesis.

The deduction follows from adding $F_{2n}{}^2$ to both sides of the result just proved.

(iv) This result cannot be deduced directly from (iii) as the nature of the expression differs in the type of subscript. Thus consider

$$F_{2n-1}{}^2+1=\left(F_{2n+1}-F_{2n}\right)^2+1=F_{2n+1}{}^2-2F_{2n+1}F_{2n}+F_{2n}{}^2+1=F_{2n+1}{}^2-2F_{2n+1}F_{2n}+F_{2n-1}F_{2n+1}$$

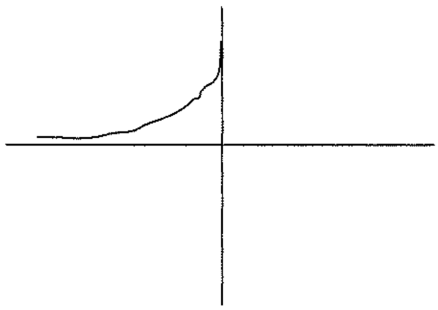
from (iii) and hence the desired result is obtained.

4.

$$y=a\sin t \Rightarrow \dot{y}=a\cos t$$

$$x=a\left(\cos t+\ln\tan\frac{t}{2}\right)\Rightarrow \dot{x}=a\left(-\sin t+\frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right)=a(-\sin t+\operatorname{cosec} t)=a\cos t\cot t$$

giving $\dfrac{dy}{dx}=\tan t$.



(y intercept a, y axis tangential to curve, x axis asymptote)

Tangent is $y-a\sin t=\tan t\left(x-a\left(\cos t+\ln\tan\frac{t}{2}\right)\right)$ giving Q as $\left(a\ln\tan\frac{t}{2},0\right)$ and thus $PQ=\sqrt{\left((a\cos t)^2+(a\sin t)^2\right)}=a$

$$\dot{y}=a\cos t\Rightarrow \ddot{y}=-a\sin t$$

$$\dot{x}=a(-\sin t+\operatorname{cosec} t)\Rightarrow \ddot{x}=a(-\cos t-\operatorname{cosec} t\cot t)$$

$$\dot{x}^2+\dot{y}^2=(a\cos t\cot t)^2+(a\cos t)^2=a^2\cot^2 t$$

$$\dot{x}\ddot{y}-\dot{y}\ddot{x}=a\cos t\cot t\times -a\sin t-a\cos t\times a(-\cos t-\operatorname{cosec} t\cot t)$$

$$=a^2\left(-\cos^2 t+\cos^2 t+\cot^2 t\right)=a^2\cot^2 t$$

giving $\rho=a\cot t$.

From the results for $\frac{dy}{dx}$ and ρ, C is

$$\left(a\left(\cos t+\ln\tan\frac{t}{2}\right)-\rho\sin t, a\sin t+\rho\cos t\right)=\left(a\ln\tan\frac{t}{2}, a\,\mathrm{cosec}\,t\right)$$

Which has the same x coordinate as Q.

5. $\frac{dr}{dx}=x\left(x^2-1\right)^{\frac{-1}{2}}=\cosh\theta$

$y=\ln r^2=2\ln r$

So $\frac{dy}{dx}=\frac{2}{r}\frac{dr}{dx}=\frac{2\cosh\theta}{r}$

$\frac{dx}{d\theta}=-\mathrm{cosech}^2\theta$ and $r=\mathrm{cosech}\theta$,

So differentiating the previous result and substituting,

$$\frac{d^2y}{dx^2}=\frac{2r\sinh\theta\frac{d\theta}{dx}-2\cosh\theta\frac{dr}{dx}}{r^2}=\frac{2\left(\mathrm{cosech}\theta\sinh\theta\times-\sinh^2\theta-\cosh\theta\cosh\theta\right)}{r^2}=-\frac{2\cosh 2\theta}{r^2}$$

Similarly,

$$\frac{d^3y}{dx^3}=-\frac{2r^2 2\sinh 2\theta\frac{d\theta}{dx}-2\cosh 2\theta\times 2r\frac{dr}{dx}}{r^4}=\frac{4}{r^4}(\sinh 2\theta+\cosh 2\theta\coth\theta)=\frac{4}{r^3}\cosh 3\theta$$

In order to hypothesise a result for $\frac{d^ny}{dx^n}$, the important thing is to appreciate that the 4 has come from 2 times exponent of r and multiple of θ.

So $\frac{d^ny}{dx^n}=2\times(-1)^{n-1}\frac{(n-1)!}{r^n}\cosh n\theta$ which may be proved by induction, the inductive differentiation step following the same pattern of working as used for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

6. $pp^*=qq^*=a^2$

and so $a^2(p-q)=qq^*p-pp^*q=-pq\left(p^*-q^*\right)$ and hence the required result.

If PQ and RS are perpendicular then $p-q=ki(r-s)$ for some real k, and thus

$p^*-q^*=-ki\left(r^*-s^*\right)$, and so $pq=-a^2\frac{p-q}{p^*-q^*}=a^2\frac{r-s}{r^*-s^*}=-rs$

For $n=3$, $B_1B_2\perp A_1A_2$ etc. $\Rightarrow a_1a_2+b_1b_2=0$ etc.

Thus $b_1{}^2=\frac{b_1b_2\times b_1b_3}{b_2b_3}=\frac{-a_1a_2\times-a_1a_3}{-a_2a_3}=-a_1{}^2$ and so $b_1=\pm ia_1$

i.e. two choices of B_1.

For $n=4$, $B_1B_2 \perp A_1A_2$ etc. $\Rightarrow a_1a_2 + b_1b_2 = 0$ etc. but this only yields 3 independent equations as e.g. $a_3a_4 + b_3b_4 = 0$ can be obtained from the other three equations by $a_3a_4 = \frac{a_2a_3 \times a_4a_1}{a_2a_1}$ etc. Hence there are arbitrarily many possible choices for B_1.

For $n > 4$, the corresponding results are as for $n = 3$ or $n = 4$ depending on whether n is odd or even.

7. (i) $u = v^{-1} \Rightarrow \frac{du}{dv} = -v^{-2}$ so $t(x) = \int_{\infty}^{\frac{1}{x}} \frac{1}{1+v^{-2}} \times -v^{-2} dv = \int_{\frac{1}{x}}^{\infty} \frac{1}{v^2+1} dv$

so $t\left(\frac{1}{x}\right) + t(x) = \int_0^{\frac{1}{x}} \frac{1}{1+u^2} du + \int_{\frac{1}{x}}^{\infty} \frac{1}{v^2+1} dv = \int_0^{\infty} \frac{1}{1+u^2} du = \frac{1}{2}p$

Letting $x = 1$ gives the desired result.

(ii) $y = \frac{u}{\sqrt{1+u^2}} \Rightarrow u = \frac{y}{\sqrt{1-y^2}}$

so $\frac{du}{dy} = \frac{(1-y^2)^{\frac{1}{2}} - y \times -y(1-y^2)^{\frac{-1}{2}}}{1-y^2} = \frac{(1-y^2)+y^2}{(1-y^2)^{\frac{3}{2}}}$ and hence the result.

Using the given substitution for u,

$$t(x) = \int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{1+\frac{y^2}{1-y^2}} \times \frac{1}{(1-y^2)^{\frac{3}{2}}} dy = \int_0^{\frac{x}{\sqrt{1+x^2}}} \frac{1}{(1-y^2)^{\frac{1}{2}}} dy = s\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Again letting $x = 1$, and using the result from part (i) gives the desired result.

(iii) $z = \frac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u} \Rightarrow u = \frac{z - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}z} \Rightarrow \frac{du}{dz} = \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2}$

Using this substitution,

$$t(x) = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{1}{1+\left(\frac{z-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}z}\right)^2} \times \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2} dz = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2 + \left(z-\frac{1}{\sqrt{3}}\right)^2} dz = \int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}} \frac{1}{1+z^2} dz$$

Letting $x = \frac{1}{\sqrt{3}}$ gives the required result.

By definition $t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+u^2} du$, by the previous result just obtained

$t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2} dz$, and from part (i) $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\sqrt{3}}^{\infty} \frac{1}{1+v^2} dv$ and so adding these three

results gives $3t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\infty} \frac{1}{1+u^2} du = \frac{1}{2}p$

8. (i) Substituting each u into the differential equation yields simultaneous equations $a(x) + xb(x) = 0$ and $e^{-x}(1 - a(x) + b(x)) = 0$ which solve to give

$a(x) = \frac{x}{1+x}$ and $b(x) = \frac{-1}{1+x}$

The general solution is $u = Ax + Be^{-x}$.

$y = \frac{1}{3u}\frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{3u^2}\left(\frac{du}{dx}\right)^2 + \frac{1}{3u}\frac{d^2u}{dx^2}$ which when substituted into equation (*), multiplied by 3u, and collected on one side gives the required result.

$u = Ax + Be^{-x} \Rightarrow \frac{du}{dx} = A - Be^{-x} \Rightarrow y = \frac{A - Be^{-x}}{3(Ax + Be^{-x})}$,

and substitution of $x = 0, y = 0$ gives $A = B$ and hence $y = \frac{1 - e^{-x}}{3(x + e^{-x})}$.

(ii) Substituting $y = \frac{1}{u}\frac{du}{dx}$ into the given equation yields

$\frac{d^2u}{dx^2} + \frac{x}{1-x}\frac{du}{dx} - \frac{1}{1-x}u = 0$ which is the equation in the first part with x replaced by $-x$

So the general solution is $u = Cx + De^{x}$

Substitution of $x = 0, y = 2$ again gives $A = B$, and hence $y = \frac{1 + e^{x}}{x + e^{x}}$

Section B: Mechanics

9. Conservation of energy leads to the equation

$2\left[\frac{1}{2}m\left(a\dot{\theta}\right)^2\right] + mk^2a^2(\theta - \alpha)^2 = mk^2a^2(\beta - \alpha)^2$ which, when simplified, and working in the variable $(\theta - \alpha)$ rather than θ can be rearranged as

$(\dot{\theta} - \alpha) = k\sqrt{\left((\beta - \alpha)^2 - (\theta - \alpha)^2\right)}$.

Separating the variables and performing the standard integral yields $\theta - \alpha = (\beta - \alpha)\sin(kt + \phi)$ (it does not matter that $(\beta - \alpha) < 0$).

The initial position from which the system is released gives $\phi = \frac{\pi}{2}$ and so

$\theta = \alpha + (\beta - \alpha)\cos kt$.

The three possibilities that can arise are that $\dot{\theta} = 0, \theta < \frac{\pi}{2}$, that $\dot{\theta} = 0, \theta = \frac{\pi}{2}$, or that $\dot{\theta} > 0, \theta = \frac{\pi}{2}$.

The first of these is SHM and has period $\frac{2\pi}{k}$, which occurs if $\alpha - (\beta - \alpha) < \frac{\pi}{2}$ i.e. if $\beta > 2\alpha - \frac{\pi}{2}$.

For the second case, oscillations do not occur. Then,

$\dot{\theta} = 0 \Rightarrow \sin kt = 0 \Rightarrow \cos kt = -1$ (not $\cos kt = 1$ as this is the initial position) and so $\frac{\pi}{2} = \alpha - (\beta - \alpha)$ i.e. $\beta = 2\alpha - \frac{\pi}{2}$.

The third case is partially SHM until $\theta = \frac{\pi}{2}$ and then the motion is reflected.

So a quarter of the period is given by $\frac{\pi}{2} = \alpha + (\beta - \alpha)\cos kt$ and hence the period is

$\frac{4}{k}\cos^{-1}\left(\frac{\frac{\pi}{2} - \alpha}{\beta - \alpha}\right)$ which occurs if $\beta < 2\alpha - \frac{\pi}{2}$.

10. Using uniform acceleration formulae with $(\ddot{x}, \ddot{y}) = (-g\sin\phi, -g\cos\phi)$, then

$(x, y) = \left(Vt\cos\theta - \frac{1}{2}gt^2\sin\phi, Vt\sin\theta - \frac{1}{2}gt^2\cos\phi\right)$.

To return on the same path $\dot{x} = 0$ when $y = 0$. So $t = \frac{V\cos\theta}{g\sin\phi} = \frac{2V\sin\theta}{g\cos\phi}$

i.e. $2\tan\phi\tan\theta = 1$

Also using $v^2 = u^2 + 2as$ in the x direction $0 = V^2\cos^2\theta - 2gR\sin\phi$

i.e. $R = \frac{V^2\cos^2\theta}{2g\sin\phi}$.

Thus

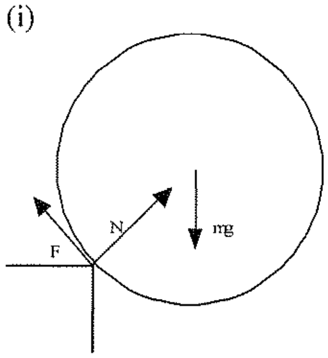
$\frac{2V^2}{gR} = 4\sin\phi\sec^2\theta = 4\sin\phi(1 + \tan^2\theta) = 4\sin\phi\left(1 + \frac{1}{4}\cot^2\phi\right) = 4\sin\phi\left(1 + \frac{1}{4}(\mathrm{cosec}^2\phi - 1)\right)$

$= 3\sin\phi + \mathrm{cosec}\phi$

Consider $y = 3x + \frac{1}{x}, x > 0$. By differentiation, this is least for $x = \frac{1}{\sqrt{3}}$.

Thus the least value of $\frac{2V^2}{gR}$ is $2\sqrt{3}$, and the largest value of R is $\frac{V^2}{\sqrt{3}g}$.

11. (i)



If the angle between mg and N is θ, then conserving energy and either differentiating the energy equation or taking moments about the point of contact yields

$$\frac{1}{2}mu^2 + mga = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta \text{ and } 0 = a\ddot{\theta} - g\sin\theta$$

Resolving in the opposite direction to F, $mg\sin\theta - F = ma\ddot{\theta}$ and so, from the second equation above, $F = 0$.

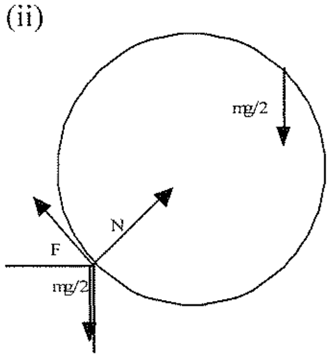
Resolving in the opposite direction to N, $mg\cos\theta - N = ma\dot{\theta}^2$,

and losing contact $N = 0$, so $a\dot{\theta}^2 = g\cos\theta$.

Thus from the energy equation $u^2 + 2ag = 3ag\cos\theta$ and so the hub has fallen

$a - a\cos\theta = a - \dfrac{u^2 + 2ag}{3g} = \dfrac{ag - u^2}{3g} > 0$, but is less than a.

(ii)



As before $\frac{1}{2}\frac{m}{2}(2u)^2 + \frac{m}{2}g(2a) = \frac{1}{2}\frac{m}{2}(2a)^2\dot{\theta}^2 + \frac{m}{2}g(2a)\cos\theta$ and $0 = 2a\ddot{\theta} - g\sin\theta$,

and $mg\sin\theta - F = \frac{m}{2}(2a)\ddot{\theta}$ so $F = \frac{1}{2}mg\sin\theta$.

Also $mg\cos\theta - N = \frac{m}{2}(2a)\dot{\theta}^2$ and so when contact is lost $N = 0$, so $a\dot{\theta}^2 = g\cos\theta$,

$u^2 + ag = 2ag\cos\theta$,

and the hub has fallen $a - a\cos\theta = a - \dfrac{u^2 + ag}{2g} = \dfrac{ag - u^2}{2g} > 0$, but is less than a.

So when $N = 0$, $\mu N = 0$, $F > 0$, but we require $F < \mu N$ not to slip, and hence slipping will certainly occur before it loses contact with the table.

Section C: Probability and Statistics

12.

$$E(N) = \sum_{i=1}^{2n-1} \frac{1}{2n-1} i = \frac{1}{2n-1} \frac{(2n-1)2n}{2} = n$$

$$E(N^2) = \sum_{i=1}^{2n-1} \frac{1}{2n-1} i^2 = \frac{1}{2n-1} \frac{(2n-1)2n(4n-1)}{6} = \frac{n(4n-1)}{3}$$

$$E(Y) = E\left(\sum_{i=1}^{N} X_i\right) = \frac{1}{2n-1} E(X_1) + \frac{1}{2n-1} E(X_1 + X_2) + \ldots = \frac{1}{2n-1}(\mu + 2\mu + 3\mu + \ldots + (2n-1)\mu)$$

$$= \frac{1}{2n-1} \frac{\mu(2n-1)2n}{2} = n\mu$$

$$E(YN) = \frac{1}{2n-1} \times 1 \times \mu + \frac{1}{2n-1} \times 2 \times 2\mu + \ldots + \frac{1}{2n-1} \times (2n-1) \times (2n-1)\mu = \frac{n(4n-1)}{3} \mu$$

and so $Cov(Y, N) = \frac{n(4n-1)}{3} \mu - n^2 \mu = \frac{1}{3} n(n-1)\mu$

$$E\left(X_i^2\right) = Var(X_i) + \left(E(X_i)\right)^2 = \sigma^2 + \mu^2$$

Also $(X_1 + X_2 + \ldots + X_r)^2 = \sum_{i=1}^{r} X_i^2 + 2\sum_{i \neq j} X_i X_j$, and so

$$E\left((X_1 + X_2 + \ldots + X_r)^2\right) = r(\sigma^2 + \mu^2) + 2\frac{r(r-1)}{2} \mu^2$$

Thus

$$E(Y^2) = \frac{1}{2n-1} \sum_{r=1}^{2n-1} \left(r(\sigma^2 + \mu^2) + 2\frac{r(r-1)}{2} \mu^2 \right) = n(\sigma^2 + \mu^2) + \frac{n(4n-1)}{3} \mu^2 - n\mu^2 = n\sigma^2 + \frac{n(4n-1)}{3} \mu^2$$

and so $Var(Y) = n\sigma^2 + \frac{n(4n-1)}{3} \mu^2 - n^2 \mu^2 = n\sigma^2 + \frac{n(n-1)}{3} \mu^2$

13. (i) $p_2(2)$ is the probability of landing in the pool for the first time on the 2nd jump starting 1.5m away which is the probability that the first jump is 1m which is p.

(ii) $u_1 = 1$

$p_2(1) = q$ and $p_2(2) = p$ so $u_2 = q + 2p = 1 + p = 2 - q$

$p_3(1) = 0$, $p_3(2) = 1 - p^2 = q(1 + p) = 2q - q^2$, and $p_3(3) = p^2 = 1 - 2q + q^2$ so

$u_3 = 2(2q - q^2) + 3(1 - 2q + q^2) = 3 - 2q + q^2$

(iii) Using the values $u_1 = 1$, $u_2 = 2 - q$, and $u_3 = 3 - 2q + q^2$, we obtain three equations:-

$A + B + C = 1$ (1)

$-Aq + B + 2C = 2 - q$ (2)

$Aq^2 + B + 3C = 3 - 2q + q^2 \quad (3)$

It makes sense to consider $(3)-(2)$ and $(2)-(1)$ to eliminate B and then subtract the resulting equations to eliminate C, and hence we find that

$(3)-2(2)+(1) \Rightarrow A\left(q^2+2q+1\right)=q^2 \Rightarrow A=\left(\frac{q}{q+1}\right)^2,$

substituting in $(2)-(1) \Rightarrow \left(\frac{q}{q+1}\right)^2\left(-q-1\right)+C=1-q \Rightarrow C=\frac{1}{1+q}$, and so

$B=\frac{q}{(q+1)^2}$.

So $u_n = \left(\frac{q}{q+1}\right)^2 (-q)^{n-1} + \frac{q}{(q+1)^2} + \frac{1}{1+q}n = \frac{(-q)^{n+1}}{(q+1)^2} + \frac{q}{(q+1)(p+2q)} + \frac{1}{p+2q}n$

For large n, the first term approaches zero, and the second term is negligible in comparison with the third for $\frac{q}{q+1} < 1 << n$

Hence $u_n \approx \frac{1}{p+2q}n$

The expected distance covered in one jump is $q+2p$ and as jumps are of integer length, to get to the pool from a distance $\left(n-\frac{1}{2}\right)m$ needs a distance n metres to be jumped and so the expected number of jumps would be $\frac{1}{p+2q}n$.

14. (i) If W is the area of the smallest circle with centre O that encloses the hole made by a single dart throw then the p.d.f. of W is given by

$$f(w) = \begin{cases} \frac{1}{\pi}, 0 \le w \le \pi \\ 0, otherwise \end{cases}$$

If X is the area of the smallest circle with centre O that encloses all the n holes made then

$P(x < X < x + \delta x) = n\left(\frac{x}{\pi}\right)^{n-1}\frac{\delta x}{\pi}$ and so $E(X) = \int_0^{\pi} x \times n\left(\frac{x}{\pi}\right)^{n-1}\frac{1}{\pi}dx = \frac{n\pi}{n+1}$.

On the other hand, if Y is the area of the smallest circle with centre O that encloses all the $(n-1)$ holes nearest to O then $P(x < Y < x + \delta x) = n(n-1)\left(\frac{x}{\pi}\right)^{n-2}\left(1-\frac{x}{\pi}\right)\frac{\delta x}{\pi}$ and

so $E(Y) = \int_0^{\pi} x \times n(n-1)\left(\left(\frac{x}{\pi}\right)^{n-2} - \left(\frac{x}{\pi}\right)^{n-1}\right)\frac{1}{\pi}dx = \frac{(n-1)\pi}{n+1}$

(ii) If Z is the area of the smallest square with centre Q that encloses all the n holes made then, in similar manner to (i)

$P(x < Z < x + \delta x) = n\left(\frac{x}{4}\right)^{n-1}\frac{\delta x}{4}$ and so $E(Z) = \int_0^{4} x \times n\left(\frac{x}{4}\right)^{n-1}\frac{1}{4}dx = \frac{4n}{n+1}$.

(iii) If we knew that the dart landed inside the circle of radius 1 centre Q when it hit the square dartboard, then the answer would be that we obtained for the circular board. But there is a non-zero probability that the dart could land in larger circles if it fell on the board outside the circle of radius 1 and hence the expected area of the smallest circle for the square dartboard is larger than that for the circular board.

Algebraically, if S is the expected area of such a circle if the dart falls outside the circle on the square board, and $E(X)$ is as in part (i),

the expected area = $\left(\frac{\pi}{4}\right)E(X)+\left(1-\frac{\pi}{4}\right)S$, where $S > E(X)$, and so this is

$$\left(1-\left(1-\frac{\pi}{4}\right)\right)E(X)+\left(1-\frac{\pi}{4}\right)S = E(X)+\left(1-\frac{\pi}{4}\right)(S-E(X)) > E(X)$$